ISRAEL JOURNAL OF MATHEMATICS, Vol. 50, No. 3, 1985

BROWNIAN MOTION NEVER INCREASES: A NEW PROOF TO A RESULT OF DVORETZKY, ERDŐS AND KAKUTANI

BY

OMER ADELMAN

Université Pierre et Marie Curie, Laboratoire de Probabilités, 4, Place Jussieu-Tour 56, 75230 Paris Cedex 05, France

ABSTRACT

Dvoretzky, Erdős and Kakutani discovered that the Brownian motion is almost surely nowhere increasing. This is proved here in a relatively easy way.

For a real function f defined on the real line $\mathbf{R}$, we say that f increases at t if, and only if, there exist a and b such that $a < t < b$ and such that if $a < s < t < u < b$, then $f(s) \leqq f(t) \leqq f(u)$.

Let X be a (continuous) one-dimensional Brownian motion defined on $\mathbf{R}$. (X can be constructed out of two independent standard Brownian motions, B and B', by letting $X_t = B_t$ for $t \geqq 0$, $= B'_{-t}$ for $t < 0$.)

Dvoretzky, Erdös and Kakutani proved in [1] the following

THEOREM$_1$. *Almost surely, X is nowhere increasing.*

Their proof is ingenious and rather difficult. Here I propose an alternative approach, converting the problem, through a series of rather obvious reductions, to a relatively easy one.

We say that a function $f: \mathbf{R} \rightarrow \mathbf{R}$ is totally increasing (or: admits a total increase) at t ($\in \mathbf{R}$) if, and only if,

$$-\infty < s < t < u < \infty \Rightarrow f(s) < f(t) < f(u).$$

For $\alpha \in \mathbf{R}$, let X^α denote the diffusion defined by

$$X_t^\alpha = X_t + \alpha t.$$

Received December 31, 1984

Let $t \in \mathbf{R}$. Assume

$$-\lim_{s\to-\infty} X_s^1 = \lim_{s\to\infty} X_s^1 = \infty,$$

as is almost surely the case. Clearly, if X increases at t, then there is some real $\alpha > 0$ such that X^α totally increases at t. (Let

$$h = \max\{\max X^1(]-\infty, t]) - X_t^1, X_t^1 - \min X^1([t, \infty[)\}.$$

Note the existence of some real $\Delta > 0$ such that $X_t^1 = \max X^1([t-\Delta, t]) = \min X^1([t, t+\Delta])$. Any $\alpha > 1 + h/\Delta$ will do.) This makes of theorem$_1$ an immediate corollary of the following

THEOREM$_2$. *Almost surely, for all real $\alpha > 0$, X^α admits no total increase.*

Since the existence of a total increase for X^α implies that of a total increase for X^β for all $\beta > \alpha$, theorem$_2$ is equivalent to the (a priori weaker)

PROPOSITION. *Let $Y \equiv X^\beta$ for some $\beta > 0$. Then, almost surely Y admits no total increase.*

Obvious symmetry arguments make us see that it is just as probable for Y to admit a total increase somewhere in $]0, \infty[$ as to admit one in $]-\infty, 7[$. Since $]0, \infty[\cup]-\infty, 7[= \mathbf{R}$, we conclude that the above proposition will be established once we prove that

(1) *almost surely, Y admits no total increase in* $]0, \infty[$.

Set

$$t_0 = m_0 = 0 \quad (= Y_0),$$

and define, recursively,

$$t_n = \sup\{t / Y_t = m_{n-1}\},$$

$$m_n = \sup Y([0, t_n]) \qquad (n = 1, 2, 3, \ldots).$$

Let

$$U = \lim_{n\to\infty} t_n.$$

Clearly, Y admits no total increase in $]0, U[$. So, in order to prove (1), it is sufficient to show that

(2) *almost surely,* $U = \infty$.

Let $\varepsilon_0 = m_1$ and define, recursively, for $n = 0, -1, -2, \dots$,

$$t_{n-1} = \sup\{t \leqq 0 \mid Y_{t_n} - Y_t = \varepsilon_{-n}\},$$

$$\varepsilon_{-n} = \sup Y([t_n, t_{n+1}]) - Y_{t_{n+1}}.$$

Let y_n denote the restriction of $Y_{t_n+\cdot} - Y_{t_n}$ to the interval $[0, t_{n+1} - t_n]$.

Observe that the sequences $(y_0, y_1, y_2, \dots)$ and $(y_0, y_{-1}, y_{-2}, \dots)$ have the same law.

Let

$$T = \lim_{n \to -\infty} t_n.$$

This make of (2) an immediate consequence of:

(3) *Almost surely,* $T = -\infty$.

Now, let

$$s_0 = 0, \qquad \delta_0 = \varepsilon_0$$

and define, recursively,

$$s_{n-1} = \sup\{t \leqq 0 \mid X_{s_n} - X_t = \delta_{-n}\} \qquad (n = 0, -1, -2, \dots),$$

$$\delta_{-n} = \sup X([s_n, s_{n+1}]) - X_{s_{n+1}} \qquad (n = -1, -2, \dots).$$

Finally, let

$$S = \lim_{n \to -\infty} s_n.$$

Assume, for a moment, that (3) is false. This implies the existence of some $t \in]-\infty, 0[$ such that $P(T > t) > 0$. Since for each $r \in]-\infty, 0[$, $Y_{|[r,0]}$ and $X_{|[r,0]}$ are equivalent in law, this entails $P(S > t) > 0$ for some $t \in]-\infty, 0[$. (Recall the analogy between the definition of T relative to $Y_{|]-\infty,0]}$ and that of S relative to $X_{|]-\infty,0]}$.)

So (3) becomes equivalent to

(4) *almost surely,* $S = -\infty$.

Observe that if $S > -\infty$, then $X_0 - X_S = \delta_0 + \delta_1 + \delta_2 + \cdots$ (X is continuous!), so $\lim_{n \to \infty} \delta_n = 0$. We realize that (4), and hence theorem$_1$, is a corollary of the following

LEMMA. *Almost surely,* $\limsup_{n \to \infty} \delta_n > 0$.

PROOF. Note that almost surely, for all n ($\in \{0,1,2,\dots\}$), $\delta_n > 0$. Then note that δ_1/δ_0, δ_2/δ_1, $\delta_3/\delta_2,\dots$ are iid. Now δ_1/δ_0 and δ_0/δ_1 have the same law. In fact, letting $\tau_x = \sup\{t \leqq 0 \mid X_t = x\}$, we see that if $\pi(.)$ is a version of $P(. \mid \sigma(\delta_0))$, then for almost all ω, for all real $a > 0$,

$$\begin{aligned}\pi(\delta_1/\delta_0 \geqq a)(\omega) &= \pi(\tau_{a\delta_0} < \tau_{-\delta_0})(\omega)\\ &= P(\tau_{a\delta_0(\omega)} < \tau_{-\delta_0(\omega)})\\ &= P(\tau_{-\delta_0(\omega)} < \tau_{\delta_0(\omega)/a})^{\dagger}\\ &= \pi(\tau_{-\delta_0} < \tau_{\delta_0/a})(\omega)\\ &= \pi(\delta_1/\delta_0 \leqq 1/a)(\omega).\end{aligned}$$

We easily deduce that, for all m and n, δ_n/δ_m and δ_m/δ_n have the same law.

Now, the event $\{\lim_{n\to\infty} \delta_n/\delta_0 = 0\}$ is a tale event for the independent sequence $(\delta_1/\delta_0, \delta_2/\delta_1, \delta_3/\delta_2,\dots)$. Its probability is thus 0 or 1. Were it 1, δ_n/δ_0 and δ_0/δ_n would both converge to zero in probability. But how can both $P(\delta_n/\delta_0 < 1)$ and $P(\delta_0/\delta_n < 1)$ tend to 1 (and be, eventually, larger than 0.6)?

So, almost surely, δ_n (just as δ_n/δ_0) does *not* converge to zero.

OUTLINE OF AN ALTERNATIVE PROOF. Observe that δ_{k+1}/δ_k are iid, each having the law of V/W, V and W being two independent exponential random variables (with the same parameter). So

$$\mathrm{E}\log\frac{\delta_{k+1}}{\delta_k} = \mathrm{E}\log V - \mathrm{E}\log W = 0.$$

This implies that, almost surely

$$\log\frac{\delta_n}{\delta_0} = \log\frac{\delta_1}{\delta_0} + \log\frac{\delta_2}{\delta_1} + \cdots + \log\frac{\delta_n}{\delta_{n-1}}$$

does *not* tend to $-\infty$.

REFERENCE

1. A. Dvoretzky, P. Erdős and S. Kakutani, *Nonincrease everywhere of the Brownian motion process*, Proc. 4th Berkeley Symp., II, 1961, pp. 103–116.

† If $x \neq 0$, then $P(\tau_{ax} < \tau_{-x})$ is independent of x (and, by the way, equals $1/(a+1)$).